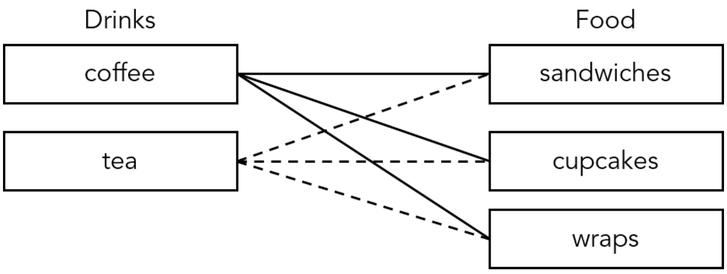
Permutations and Combinations

- list and count the numbers of ways an event can occur
- use the fundamental counting principle (i.e. multiplication principle)
- solve simple problems and proves results using the pigeonhole principle
  − if there are n pigeonholes and $n + 1$ pigeons, then at least one pigeonhole must hold 2 or more pigeons
  − generalise to: If n pigeons are sitting in k pigeonholes, when $n > k$, then there is at least one pigeonhole with at least $\frac{n}{k}$ pigeons in it
  − prove the pigeonhole principle
- understand and use combinations to solve problems
  − understand and use the notations $\binom{n}{r}$ and ${}^nC_r = \frac{n!}{r!(n-r)!}$

Combinations

THE COUNTING PRINCIPLE

- If there are x ways to perform one task and y ways to perform another task, then there are a total of $x \times y$ ways to perform both tasks.
  - This is called the multiplication principle.
- As a very basic example to comprehend this idea, consider a student at a school canteen with the option of:
  - Drinks: coffee, tea
  - Food: sandwiches, cupcakes, wraps



∴ Total of $2 \times 3 = 6$ choices of varying combinations.

- Formally the multiplication principle is as follows:

> Suppose that we perform r experiments such that the k^{th} experiment has n_k possible outcomes, for $k = 1, 2, \ldots r$. Then there are a total of $n_1 \times n_2 \times n_3 \times \ldots \times n_r$ possible outcomes for the sequence of r experiments.

WORKED EXAMPLE

a) Suppose that Alan wants to purchase a tablet computer. He can choose either a large or a small screen; a $64GB, 128GB$ or $256GB$ storage capacity, and a black or white cover. How many different options does he have?

b) To create a password for a computer account, a set of rules must be followed. The rule is that the password must consist of two lowercase letters (a to z) followed by one capital letter (A to Z) followed by four digits ($0, 1, \ldots 9$). For example, the following is a valid password:

$$ejT3018$$

Find the total number of possible passwords, N.

SOLUTIONS

a) There are a total of 2 choices for screen size, 3 options for storage capacity and 2 choices for the colour of the cover. Hence, following the multiplication rule, the total number of options is given by:

$$2 \times 3 \times 2 = 12 \text{ options}$$

b) There are a total of 2×26 options for the two lowercase letters, 26 options for the capital letter and 4×10 options for the four digits. Hence, as per the multiplication rule, the total number of options is given by:

$$(2 \times 26) \times (26) \times (4 \times 10) = 54080 \text{ options}$$

THE PIGEONHOLE PRINCIPLE

- The pigeonhole principle is as follows:

> **If $n+1$ pigeons are put into n pigeonholes, then some hole must contain at least 2 pigeons.**

- Although pigeons and pigeonholes are the syntax used, a more general placeholder for each term can be used instead:
  - Pigeons can be replaced by 'items'.
  - Pigeonholes can be replaced by 'containers'.
- Take for example a basic scenario of 13 people sitting inside a classroom and we were performing a survey of their birth month. It follows that there are:
  - 13 items (i.e. people)
  - 12 containers (i.e. months)
- Therefore, according to the pigeon principle, even at the most spread out distribution of one student born per month, there will be 2 students that share a birthday on the same month.
- Although the pigeonhole principle seems very simple and intuitive, there are many real-life applications that were not immediately obvious prior to understanding this rule:
  - If there are 3 pairs of socks in a drawer, then picking 4 socks guarantees that at least a pair is chosen
  - A committee can be formed by choosing $10 \times$ Year 7 or $8 \times$ Year 8 or $5 \times$Year 9 students. Therefore, the minimum number of students randomly chosen to ensure a committee is formed is $(9+7+4)+1 = 21$.

- The formal definition for the pigeonhole principle is:

> Given m items and n containers, if $m > n$,
> there is at least one container with $\left\lceil \frac{m}{n} \right\rceil$ items.

  - The $\lceil\ \rceil$ symbol is called the 'ceiling' and refers to the integer when the value inside is rounded up.
  - If we apply the pigeonhole rule to the example above about students and their birth months, we can say that there is at least one month with $\left\lceil \frac{13}{12} \right\rceil$ students that have a birthday.
  - Here, $\left\lceil \frac{13}{12} \right\rceil = 1.0833 \ldots = 2$

WORKED EXAMPLE

a) A box contains 12 white, 9 blue and 11 red disks. What is the minimum number of disks that must be randomly chosen to ensure that you have 5 disks of the same colour?

b) Write a formula for the number of objects to be randomly chosen from a container that holds q_1 objects O_1, q_2 objects $O_2, \ldots, q_n$ objects O_n so that O_i, where $i = 1, 2, \ldots, n$, has q_i objects.

SOLUTIONS

a) At most, you will pick 4 of each colour. The one thereafter will be the 5^{th} disk of the same colour. Hence the total is: $(4 \times 3) + 1 = 13$ disks.

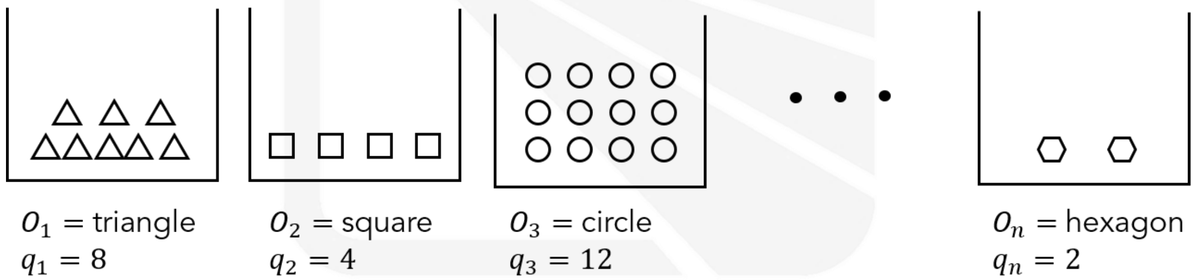
b) At first, this question can appear very confusing. Let's first realise that q stands for quantity and O stands for object.

The question is asking about the minimum number of randomly chosen objects such that O_i has q_i objects. In other words, for a randomly chosen object O_i we choose ALL of it from the container.

In this case, we must pick out $q_i - 1$ of each object, where $i = 1, 2, \dots, n$, and the object after that will be the final object required to ensure we have chosen the last one for any object, fulfilling the specification.

Hence the answer is: $[(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1)] + 1 = (\sum_{i=1}^{n} q_i - 1) + 1$

To better comprehend this question, let's consider the diagram below of containers, each holding a unique object.



If we were to randomly pick out items from each container, then the minimum required such that one of the containers will be certain of being empty is to pick out $q_i - 1$ from each container where $i = 1, 2, \dots, n$.

At this point, there will only be 1 object left in this container. Hence, we only need to pick one more so that we will have q_i of object O_i. This is summarised in the formula above.

COMBINATIONS

- A combination is a collection of unique elements from a given set, where order is not taken into account. Each element only appears once in a combination, it cannot be repeated.
  - How many ways can 3 coloured pencils be selected from a pack of 5?
  - How many ways can 4 students be chosen from a class of 12 to form a team?
- Each selection in the examples above are a combination. In general terms, the number of different combinations of r objects from n distinct objects is:

$$^{n}C_{r} = \binom{n}{r}$$

$$= \frac{n!}{r!\,(n-r)!}$$

> A factorial can be 'unrolled' by writing it as a factorial of the adjacent term:
>
> $$n! = n(n-1)!$$
>
> By definition, $0! = 1$

  - where $n! = 1 \times 2 \times 3 \times \ldots \times n$ ('n' factorial).
  - $^{n}C_{r}$ is read as: from a set of 'n' distinct objects we choose 'r' lots of them.

WORKED EXAMPLES

a) By listing out all possibilities, find out how many ways we can choose 3 letters from this set of 4 letters $\{A, B, C, D\}$ and verify it with the formula $^{n}C_{r} = \frac{n!}{r!(n-r)!}$.

b) (Selecting groups based on conditional criteria) A bucket contains the following marbles: 4 red, 3 blue, 4 green and 3 yellow making a total of 14 marbles. Each marble is labeled with a number so they can be distinguished (i.e. all unique objects).

i. How many sets of 4 marbles are possible?
ii. How many sets of 4 are there such that each one is a different colour?
iii. How many sets of 4 are there in which at least 2 are red?

SOLUTIONS

a) The possibilities are: $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, $\{B, C, D\}$, recalling that order does not matter. Hence, there should be 4 ways of choosing 3 letters from the set.

We will now verify it using the formula for $n = 4$ and $r = 3$:

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$= \frac{4!}{3!\,(4-3)!}$$

$$= \frac{\cancel{1 \times 2 \times 3} \times 4}{(\cancel{1 \times 2 \times 3})(1)}$$

$$= 4$$

b)

i.

$$^{14}C_{4} = \frac{14!}{4!\,(14-4)!}$$

$$= 1001$$

ii.

$^{4}C_{1}$(red) × $^{3}C_{1}$(blue) × $^{4}C_{1}$(green) × $^{3}C_{1}$(yellow) $= 4 \times 3 \times 4 \times 3$
$= 144$

iii. To answer this question, we must split it into 3 cases:

2 red out of 4 2 *not red* out of 10	3 red out of 4 1 *not red* out of 10	4 red out of 4 0 *not red* out of 10
$^{4}C_{2} \times {}^{10}C_{2} = 270$	$^{4}C_{3} \times {}^{10}C_{1} = 40$	$^{4}C_{4} = 1$

Hence the total number of ways is $270 + 40 + 1 = 311$