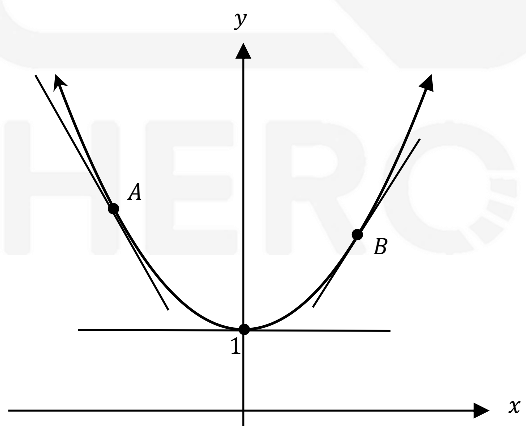
2. Difference Quotients

- describe the behaviour of a function and its tangent at a point, using language including increasing, decreasing, constant, stationary, increasing at an increasing rate

Increasing and Decreasing Functions:

- A function is INCREASING at x if the gradient of the tangent is positive
- A function is DECREASING at x if the gradient of the tangent is positive
- A function is STATIONARY at x if the tangent is horizontal i.e the gradient is zero
- A function is INCREASING at an INCREASING RATE if the tangent becomes steeper as we move from left to right

  EXAMPLE: Below is a graph of the parabola $y = f(x) = x^2 + 1$

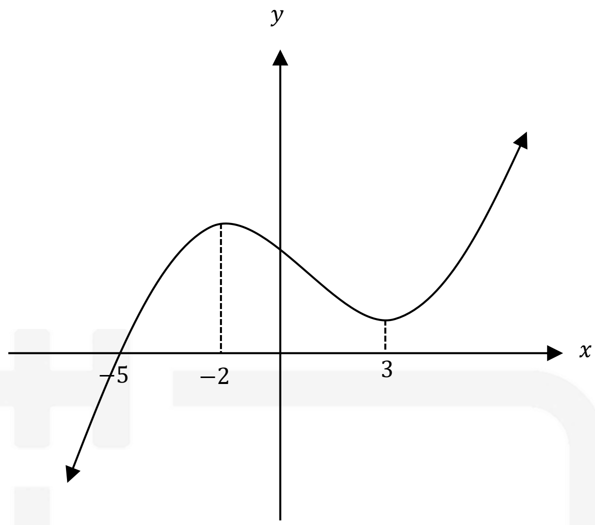


$f(x)$ is increasing at an increasing rate at point B because the gradient of its tangent is positive and the tangents for points right of B are steeper.

$f(x)$ is decreasing at point A because the gradient of its tangent is negative.

$f(x)$ is stationary at $(0, 1)$ because its tangent is horizontal

WORKED EXAMPLE: Below is a graph of some function $y = f(x)$.



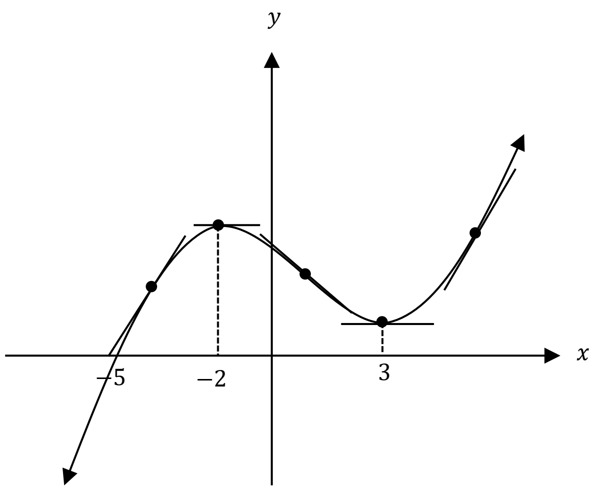
a) Where is $f(x)$ increasing?

b) Where is $f(x)$ decreasing?

c) Where is $f(x)$ stationary?

d) Where is $f(x)$ increasing at an increasing rate?

It can be helpful to draw multiple tangent lines on the graph at different points to figure out whether the gradient of the tangents are positive or negative, or zero.



a) The gradients (or slopes) of the tangents are positive for any x less than -2 or greater than 3.

Hence, $f(x)$ is increasing when $x < -2$ or $x > 3$

b) The gradients of the tangents are negative between the two 'humps' or turning points of the graph.

$f(x)$ is decreasing when $-2 < x < 3$

c) The tangents are horizontal with gradient $\mathbf{0}$ at the two turning points of the graph.

$f(x)$ is stationary when $x = -2$ or $x = 3$

d) The tangents of $f(x)$ have positive gradient and become steeper when $x > 3$

(Note how even though $f(x)$ is increasing for $x < -2$, the tangents become flatter, so it is not increasing at an increasing rate here.)

$f(x)$ is increasing at an increasing rate for $x > 3$

- interpret and use the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change of $f(x)$ or the gradient of a chord or secant of the graph $y = f(x)$

Difference Quotients

- Suppose $f(x)$ is a function and x_1 and x_2 are any two x-coordinates. The gradient of the secant through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is

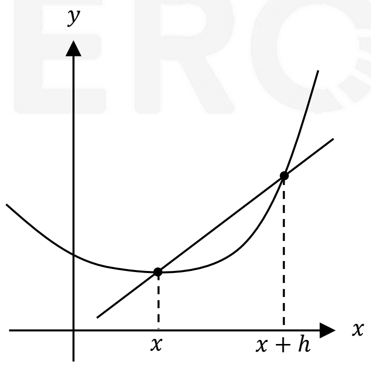
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- This is number is called a DIFFERENCE QUOTIENT because it consists of one difference divided by another

- The gradient of the secant (see diagram below) can also be written in terms of x and a positive number h by letting $x_1 = x$ and $x_2 = x + h$

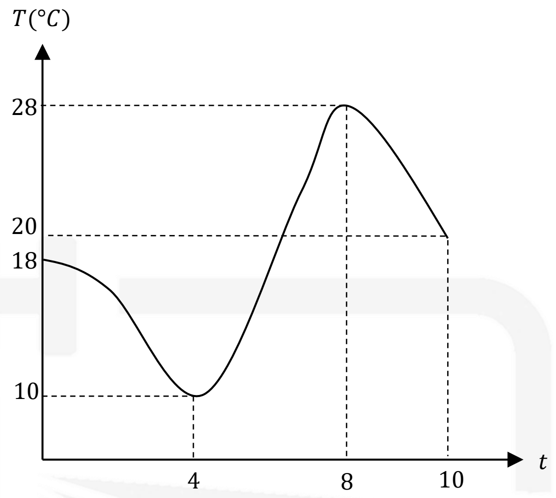
$$m = \frac{f(x+h) - f(x)}{x + h - x}$$

$$m = \frac{f(x+h) - f(x)}{h}$$



- This gradient is also equal to the AVERAGE RATE OF CHANGE of the function $f(x)$ between x_1 and x_2. It is the hypothetical rate of change of $f(x)$ between x_1 and x_2 if $f(x)$ was a straight line between x_1 and x_2

WORKED EXAMPLE: The graph below shows the air temperature for a town where t is the time in hours after measurements were began in the morning.



You may ignore units for the questions below

a) Find the average rate of change in air temperature between $t = 0$ and $t = 4$

b) Find the average rate of change in air temperature between $t = 8$ and $t = 10$

a) Let $r =$ average rate of temperature.

$$r = \frac{18 - 10}{4 - 0}$$

$$= 2$$

b) Let $r =$ average rate of temperature.

$$r = \frac{20 - 28}{8 - 10}$$

$$= -4$$

- interpret the meaning of the gradient of a function in a variety of contexts, for example on distance–time or velocity–time graphs

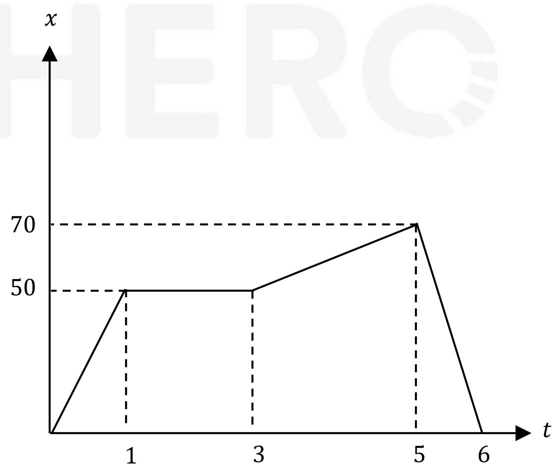
Applications of the Gradient

- If $f(t)$ is a function of time, then the gradient of a function is a rate of change.

- Two common examples of analysing rates of change (i.e the gradient) involve displacement-time graphs and velocity-time graphs. At some time t,
  - The gradient of a displacement-time graph at t is the VELOCITY at t
  - The gradient of a velocity-time graph at t is the ACCELERATION at t

WORKED EXAMPLE: A milkman travels 50 km and drops off his first delivery before taking a nap and staying put. He then travels 20 km to drop off his next delivery before immediately turning around and heading home.

Below is the displacement-time graph of his journey where t is his time in hours.



a) Find the milkman's initial velocity in kilometres per hour.

b) Find the milkman's velocity at $t = 5.5$

c) How long does the milkman nap?

a) The milkman's velocity at any time can be calculated by finding the gradient of the displacement time graph at that time.

The function is a straight line between $t = 0$ and $t = 1$, so the velocity (gradient) is the same between these two times.

$$v = m = \frac{50 - 0}{1 - 0}$$

$$= 50 \text{ km/h}$$

b) The function is a straight line between $t = 5$ and $t = 6$, so the velocity is given by

$$v = \frac{0 - 70}{6 - 5}$$

$$= -70 \text{ km/h}$$

Note how the velocity is negative because he is heading in the opposite direction to when he started.

c) The milkman is napping when the graph is constant i.e when the gradient and hence velocity is 0.

The gradient is 0 when the function is a horizontal line, i.e between $t = 1$ and $t = 3$.

Hence the milkman naps for 2 hours.